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M E X I C O

A Bayesian Model of Voting in Juries*

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Abstract

We take a game-theoretic approach to the analysis of juries by modelling voting as a game of incomplete information. Rather than the usual assumption of two possible signals (one indicating guilt, the other innocence), we allow jurors to perceive a full spectrum of signals. Given any voting rule requiring a fixed fraction of votes to convict, we characterize the unique symmetric equilibrium of the game, and we consider the possibility of asymmetric equilibria: we give a condition under which no asymmetric equilibria exist and show that, without it, asymmetric equilibria may exist. We offer a condition under which unanimity rule exhibits a bias toward convicting the innocent, regardless of the size of the jury, and we exhibit an example showing this bias can be reversed. And we prove a "jury theorem" for our general model: as the size of the jury increases, the probability of a mistaken judgment goes to zero for every voting rule, *except* unanimity rule; for unanimity rule, we give a condition under which the probability of a mistake is bounded strictly above zero, and we show that, without this condition, the probability of a mistake may go to zero.

1 Introduction

Consider a group of decision-makers who must choose one of two alternatives. Voters agree on the overall objective, but, on the basis of differential information, they may disagree on which alternative best achieves that goal. Some examples are:

- A jury deciding whether to convict or acquit a defendant. Jurors agree about the desirability of acquitting an innocent and convicting a guilty defendant, but they have different opinions about whether the defendant is innocent or guilty.
- The board of directors of a company deciding whether to approve a new investment project. All members of the board agree on the desirability of higher profits, but they disagree about whether the project is profitable.
- A group of medical experts deciding on a treatment for a patient. The common objective is the patient's health, but there is disagreement about the best procedure for the patient.

Though such examples are necessarily special, in that they presume a common objective shared among the decision-makers, they have provided a useful benchmark for the investigation of information aggregation in collective decision-making.

We follow an old literature on information aggregation in elections by focusing on the jury example. The literature traces back to Condorcet's (1785) jury theorem, which asserts that, under majority voting, large electorates should reach correct decisions with very high probability (cf. Miller (1986), Grofman and Feld (1988), Young (1988), Ladha (1992), Berg (1993)). It is traditionally assumed that each voter simply behaves "naively," as if deciding the outcome alone, but Austen-Smith and Banks (1996) observed that, given naive behavior on the part of jurors, some may have an incentive to vote "strategically." In other words, naive behavior does not generally constitute an equilibrium. The issue of strategic voting under incomplete information was also taken up by Feddersen and Pesendorfer (1996, 1997). We follow subsequent papers in analyzing voting in juries as a Bayesian game in which the jurors' opinions of guilt or innocence, i.e., their "signals," are private information.

Several versions of the jury theorem under strategic voting have been offered. Myerson (1998) introduces uncertainty about the size of the electorate and considers a countable set of players' types (signals), where the number of voters receiving any given signal is drawn from a Poisson distribution, the mean of which depends on guilt or innocence. He proves the existence of a sequence of equilibria that generate the results of Condorcet's jury theorem as the expected number of jurors goes to infinity. Wit (1998) shows that, in the Austen-Smith and Banks model, the non-trivial equilibria of the voting game corroborate Condorcet's jury theorem. More generally, McClennan (1998) proves that, given any voting rule, there is at least one equilibrium that maximizes the *ex ante* payoffs of jurors over the class of all symmetric strategy profiles. Since naive voting is a symmetric strategy profile, each juror's payoff from that optimal equilibrium strategy profile is at least equal to the payoff from voting naively. As a consequence, if Condorcet's jury theorem holds under naive voting, then it will also hold in the optimal equilibrium. If there are no other equilibria, then, of course, the jury theorem is completely robust to the strategic incentives of the jurors.

Feddersen and Pesendorfer (1998) analyze a model in which there are two possible signals, one indicating guilt and the other innocence. Given any voting rule requiring a fixed fraction of votes to convict, they are able to explicitly solve for the unique symmetric, responsive Bayesian equilibrium of the voting game. They show that a jury theorem holds for all voting rules other than unanimity: as the size of the jury increases, the probability of a mistaken judgment goes to zero for all voting rules except unanimity; in that case, the probability of a mistake is bounded strictly above zero. Feddersen and Pesendorfer (1998) also give an example comparing different voting rules for a fixed jury size: there, the probability of convicting an innocent defendant under unanimity rule is greater than the probability under majority or any other supermajority rule. McKelvey and Palfrey (1998) offer experimental results on the "binary signal" model roughly consistent with the equilibrium predictions.

We depart from the previous literature on juries by assuming that the signals representing the jurors' opinions of guilt or innocence are drawn from continuous, rather than discrete (usually binary), distributions. This is meant to capture the fact that a juror's opinion of the evidence against the defendant, the case made by the prosecutor, etc., may reflect a very rich spectrum of possibilities — possibilities that cannot be summarized by a dichotomous signal merely indicating guilt or innocence. We impose very few restrictions on the distributions of signals, and we obtain a continuous analogue of the binary signal model as a special case. Unlike Feddersen and Pesendorfer (1997), we confine our attention to the case in which the objectives of the jurors are perfectly aligned. Within this framework, we investigate the issues of equilibrium existence and uniqueness, the generalization of Feddersen and Pesendorfer's (1998) jury theorem, as well as the extension of their results on the inferiority of unanimity rule.

We offer three sets of results. First, we establish the existence of a symmetric, responsive equilibrium characterized by a cutoff signal: jurors who get signals indicating a higher likelihood of guilt vote for convicting the defendant while those who get signals indicating a lower likelihood vote for acquittal. The equilibrium is unique within that class. Moreover, under a strict monotone likelihood ratio condition, all equilibria are cutoff equilibria; as a consequence, our uniqueness result extends to the class of all symmetric, responsive strategy profiles, even allowing for mixed strategies. With McClennan's (1998) result, this implies that the equilibrium is optimal: it maximizes the jurors' *ex ante* payoffs over the set of symmetric strategy profiles. An undesirable artifact of the binary signal model of Feddersen and Pesendorfer (1998), and of the continuous version we consider, is that the typical juror who votes to acquit must use mixed strategies and is, therefore, indifferent between voting to acquit or convict. When continuous distributions are allowed for, that need no longer be the case. Indeed, under the strict monotone likelihood ratio condition, all equilibria are essentially *strict:* only a juror who receives the cutoff signal (a zero probability event) is indifferent to which vote he casts.

Second, we turn our attention to unanimity rule and give a sufficient condition for the symmetric, responsive cutoff equilibrium to be unique within the class of all (possibly asymmetric) responsive strategy profiles. We then give a sufficient condition for unanimity rule to exhibit a bias toward convicting the innocent, *independently* of the size of the jury. This condition is met in our continuous version of the binary signal model, with the following implication: unanimity rule leads to a higher probability a convicted defendant is innocent than does majority or any other supermajority rule. Thus, we find that Feddersen and Pesendorfer's (1998) fixed jury size example generalizes within the binary signal model. However, it does not generalize completely: we provide an example of a continuous signal model in which unanimity rule performs better than simple majority rule. We then we turn to the asymptotic behavior of unanimity rule and find, in contrast to Feddersen and Pesendorfer (1988), that our conclusions depend on the structure of information in the model: the probability of making a mistake is bounded strictly above zero if the likelihood of innocence is bounded over the interval of possible signals; otherwise, as illustrated by an example, it is possible that the probability of a mistaken judgment goes to zero as the size of the jury increases.

Third, we obtain a jury theorem for the general continuous signal model: for all voting rules other than unanimity, the probability of a mistaken judgment goes to zero as the size of the jury increases. The asymptotic efficiency of all non-unanimous voting rules is fully general, and not merely an artifact of the binary signal model. Under stronger informational assumptions than those of our model, but still assuming continuous signals, Meirowitz (1999) establishes existence and uniqueness of symmetric equilibria and also proves a jury theorem for non-unanimous voting rules. As he notes, an implication is that large juries perform better than single jurors under strategic voting, a result obtained by Condorcet (1985) under naive voting.

We leave many important issues untouched. One is the extension of our results to situations in which the jurors' preferences are not perfectly aligned, a necessary step in order to use the continuous signal model in other voting contexts. Under slightly stronger informational assumptions than ours, Li, Rosen, and Suen (1999) take up this issue in a continuous signal model with two jurors. Some other issues are: correlation among the jurors' signals, multiple "states," and multiple alternatives. Furthermore, we have not considered the optimality of different voting rules, an important but apparently difficult mechanism design issue. Finally, there is the possibility of limited communication among jurors. As stressed by Coughlan (1997), a single nonbinding "straw vote" is enough to allow jurors to share all their information in the binary signal model, thus eliminating the strategic aspects of voting in a common preference environment. In general, a finite number of “straw votes” is enough to allow jurors to share all their information if the distribution of signals is discrete. But, when the opinions of jurors can reflect subtle nuances of trials, a continuous distribution of signals seems better suited to model the difficulties associated with limited communication.

2 Preliminaries

We consider $n \geq 2$ jurors who must decide whether to convict or acquit a defendant. The defendant is either innocent, I, or guilty, G, with probabilities $P(I)$ and $P(G)$. Each juror i receives a real-valued signal s_i distributed according to $F(\cdot|I)$ or $F(\cdot|G)$, depending on whether the defendant is innocent or guilty. Conditional on the state, the signals of the jurors are drawn independently. After receiving their signals, which are private information, the jurors simultaneously vote to convict or acquit. Once the votes are tallied, the defendant's fate is determined by an anonymous, monotonic decision rule, i.e., there is some integer, k, such that the defendant is convicted, C, if k or more jurors vote to convict and acquitted, A, otherwise.

We assume the jurors have a common preference to convict the guilty and acquit the innocent. We assume that these outcomes are equally desirable and normalize the jurors' payoffs in those cases to $u(C|G) = u(A|I) = 0$. In the cases of convicting the innocent or acquitting the guilty, the jurors receive negative payoffs $u(C|I)$ and $u(A|G)$. In effect, the *ex ante* cost of conviction is $u(C|I)P(I)$, and the cost of acquittal is $u(A|G)P(G)$. We use

$$\rho \;=\; \frac{u(A|G)}{u(C|I)}\frac{P(G)}{P(I)}$$

to denote the relative *ex ante* cost of acquittal.

A strategy for juror i is a measurable mapping $\sigma_i\colon \Re \to [0,1]$, where $\sigma_i(s_i)$ is the probability that the juror votes to convict after receiving signal s_i. The probability that i votes to convict, conditional on the defendant being innocent, is

$$\int \sigma_i(s)\, \mu_I(ds),$$

where μ_I is the probability measure induced by $F(\cdot|I)$. The probability that i votes to convict conditional on guilt is identical, except that μ_G, the probability measure induced by $F(\cdot|G)$, is used. Probabilities of acquittal are written similarly, but integrating $1-\sigma_i$ rather than σ_i.

A profile of strategies is denoted $\sigma = (\sigma_1, \dots, \sigma_n)$. Given σ, the probability that the defendant is convicted conditional on being innocent, denoted $P_\sigma(C|I)$, is

$$\sum_{\substack{M\subseteq N\\ |M|\geq k}} \left[\Pi_{j\in M}\left(\int \sigma_j(s)\,\mu_I(ds)\right)\Pi_{j\notin M}\left(\int [1-\sigma_j(s)]\,\mu_I(ds)\right)\right].$$

The probability that the defendant is acquitted conditional on being guilty, denoted $P_\sigma(A|G)$, is

$$\sum_{\substack{M\subseteq N\\ |M|<k}} \left[\Pi_{j\in M}\left(\int \sigma_j(s)\,\mu_G(ds)\right)\Pi_{j\notin M}\left(\int [1-\sigma_j(s)]\,\mu_G(ds)\right)\right].$$

The *ex ante* payoff of a juror is

$$u(C|I)P_\sigma(C|I)P(I) + u(A|G)P_\sigma(A|G)P(G).$$

Let σ_{-i} represent the strategies of jurors other than i. The probabilities that i is pivotal conditional on innocence and guilt, $P_{\sigma_{-i}}(piv|I)$ and $P_{\sigma_{-i}}(piv|G)$, are defined as

$$\sum_{\substack{M\subseteq N\\ |M|=k-1\\ i\notin M}} \left[\Pi_{j\in M}\left(\int \sigma_j(s)\,\mu_I(ds)\right)\cdot\Pi_{\substack{j\notin M\\ j\neq i}}\left(\int [1-\sigma_j(s)]\,\mu_I(ds)\right)\right]$$

and

$$\sum_{\substack{M\subseteq N\\ |M|=k-1\\ i\notin M}} \left[\Pi_{j\in M}\left(\int \sigma_j(s)\,\mu_G(ds)\right)\cdot\Pi_{\substack{j\notin M\\ j\neq i}}\left(\int [1-\sigma_j(s)]\,\mu_G(ds)\right)\right],$$

respectively.

Using the above definitions, we can obtain an expression for a juror's payoff in terms of his probability of being pivotal. It translates to our framework the insight from the literature on strategic voting that a voter should condition his vote on being pivotal, as this is the only event where his vote might affect his payoff.

Proposition 1 *Given σ_{-i}, the ex ante payoff to juror i from σ_i is an affine transformation of*

$$\left(\int \sigma_i(s)\,\mu_I(ds)\right) u(C|I)P_{\sigma_{-i}}(piv|I)P(I) \\ - \left(\int \sigma_i(s)\,\mu_G(ds)\right) u(A|G)P_{\sigma_{-i}}(piv|G)P(G).$$

The proof of this (and other auxiliary results) is found in the appendix.

An *equilibrium* is a profile σ such that, for every juror i and every σ_i',

$$\begin{aligned} & u(C|I)P_\sigma(C|I)P(I) + u(A|G)P_\sigma(A|G)P(G) \\ \geq\ & u(C|I)P_{\sigma_i',\sigma_{-i}}(C|I)P(I) + u(A|G)P_{\sigma_i',\sigma_{-i}}(A|G)P(G). \end{aligned}$$

A *responsive equilibrium* is an equilibrium σ such that each σ_i is *responsive:*

$$0 < \int \sigma_i(s)\,\mu_G(ds) < 1 \quad \text{and} \quad 0 < \int \sigma_i(s)\,\mu_I(ds) < 1.$$

There are always unresponsive equilibria: if $k < n$, it is an equilibrium for the jurors to convict regardless of their signals; if $k > 1$, it is an equilibrium acquit regardless of signal. A *cutoff equilibrium* is a pure strategy equilibrium σ such that each σ_i is a *cutoff strategy:* there is some $\overline{s}_i \in [-\infty, \infty]$ such that

$$\sigma_i(s) \quad = \quad \begin{cases} 1 & \text{if } s > \overline{s}_i \\ 0 & \text{if } s < \overline{s}_i \end{cases}$$

for all $s \in \Re$. The cutoff strategy associated with a given cutoff is unique up to the behavior of the juror upon receiving the cutoff signal. Given assumption (A1), below, this is a zero-probability event, and we will not distinguish between cutoff strategies that differ only at the cutoff.

In what follows we maintain several assumptions on $F(\cdot|I)$ and $F(\cdot|G)$ that, as we will see, enable us to restrict our attention to cutoff equilibria.

(A1) *The distribution functions are absolutely continuous with respect to Lebesgue measure and have piecewise continuous densities* $f(\cdot|I)$ *and* $f(\cdot|G)$.

This assumption implies that the distribution functions are differentiable at all but a finite number of points. We will use S^d to denote the subset of signals in S on which $F(\cdot|I)$ and $F(\cdot|G)$ are both differentiable.

(A2) *The densities have common support,* $S = (\underline{S}, \overline{S})$, *where* $\underline{S}, \overline{S} \in [-\infty, \infty]$*: in particular,* $f(s|I) > 0$ *and* $f(s|G) > 0$ *for all* $s \in S$.

The latter implies that μ_I and μ_G have the same sets of measure zero. The terms "μ_I-a.e." and "μ_G-a.e." are thus synonymous, so we can use "a.e." without ambiguity. Note that a cutoff strategy given by $\overline{s}_i$ is responsive if and only if $\overline{s}_i \in S$.

(A3) *The likelihood ratio,* $f(s|I)/f(s|G)$, *is weakly decreasing on* S.

This assumption is standard and amounts to assuming that higher signals are stronger (or at least not weaker) indications of guilt. Sometimes we will want a stronger condition to hold locally: we will say that the likelihood ratio is *locally strictly decreasing at* $x \in S$ if there is an open set containing x on which the likelihood ratio is strictly decreasing.

(A4) $$\lim_{s \downarrow \underline{S}} \frac{f(s|I)}{f(s|G)} > \rho > \lim_{s \uparrow \overline{S}} \frac{f(s|I)}{f(s|G)}.$$

As we will see, a juror who behaves "naively" (i.e., as if his vote alone determines the outcome) after receiving signal s would prefer to convict if $f(s|I)/f(s|G) > \rho$ and would prefer to acquit if $f(s|I)/f(s|G) < \rho$. Thus, (A4) implies that there must be a signal low enough to induce a naive juror to acquit, and a signal high enough to induce him to convict.

Lemma 0, stated in the appendix, establishes some implications of (A1)–(A4). Among those that are well-known, $F(\cdot|G)$ exhibits (strict) first order stochastic dominance over $F(\cdot|I)$, and the ratios

$$\frac{1 - F(s|I)}{1 - F(s|G)} \quad \text{and} \quad \frac{F(s|I)}{F(s|G)}$$

are weakly decreasing.

3 Symmetric Equilibria

Consider any profile σ of responsive strategies and any juror i. Since the strategies are responsive, $P_{\sigma_{-i}}(piv|G)$ and $P_{\sigma_{-i}}(piv|I)$ are positive. Hence, under our assumptions the expression

$$J(\sigma_{-i}, s) \quad = \quad \frac{P_{\sigma_{-i}}(piv|I)}{P_{\sigma_{-i}}(piv|G)} \frac{f(s|I)}{f(s|G)} - \rho$$

is well-defined on S. Moreover, for fixed σ_{-i}, it is weakly decreasing in its second argument by (A3). Note that $J(\sigma_{-i}, s) > 0$ if and only if

$$u(C|I)P_{\sigma_{-i}}(piv|I)f(s|I)P(I) \quad < \quad u(A|G)P_{\sigma_{-i}}(piv|G)f(s|G)P(G).$$

That is, $J(\sigma_{-i}, s) > 0$ if and only if a juror's expected payoff from voting to convict is less than the expected payoff from voting to acquit, conditional on receiving signal s and on the strategies of others. Hence, as shown in the following lemma, jurors will be inclined to acquit when J is positive and to convict when it is negative. In contrast, a naive juror would behave as if the terms $P_{\sigma_{-i}}(piv|I)$ and $P_{\sigma_{-i}}(piv|G)$ were equal to one, voting to acquit if

$$u(C|I)f(s|I)P(I) \quad < \quad u(A|G)f(s|G)P(G)$$

and to convict if the inequality is reversed.

Lemma 1 *Given responsive strategies σ_{-i} for jurors other than i, a strategy σ_i is a best response for i if and only if it satisfies the following a.e.:*

$$\sigma_i(s) \quad = \quad \begin{cases} 1 & \text{if } J(\sigma_{-i}, s) < 0 \\ 0 & \text{if } J(\sigma_{-i}, s) > 0. \end{cases}$$

If the likelihood ratio is locally strictly decreasing at $\inf\{s \in S \mid J(\sigma_{-i}, s) \leq 0\}$, *$\sigma_i$ is a best response for i if and only if it is equivalent a.e. to the following cutoff strategy $\tilde{\sigma}_i$:*

$$\tilde{\sigma}_i(s) \quad = \quad \begin{cases} 1 & \text{if } J(\sigma_{-i}, s) \leq 0 \\ 0 & \text{else.} \end{cases} \qquad \square$$

Since J is weakly decreasing in its second argument, an implication of the first part of the preceding lemma is that jurors always have best response cutoff strategies. Hence, if a profile of strategies is an equilibrium when jurors are restricted to cutoff strategies, it will be an equilibrium of the unrestricted game. From the second part of the lemma, if the likelihood ratio is strictly decreasing, then *all* best responses for a juror are equivalent to cutoff strategies, regardless of the strategies of others, and *all* equilibria are equivalent to cutoff equilibria.

When all jurors other than i use the same cutoff strategy, given by cutoff $\overline{s}$, we will write $J(\overline{s}, s)$ for $J(\sigma_{-i}, s)$. That is,

$$J(\overline{s}, s) \quad = \quad \left(\frac{1 - F(\overline{s}|I)}{1 - F(\overline{s}|G)}\right)^{k-1} \left(\frac{F(\overline{s}|I)}{F(\overline{s}|G)}\right)^{n-k} \frac{f(s|I)}{f(s|G)} - \rho.$$

We will often focus on symmetric profiles of cutoff strategies, in which case we view J as a mapping defined on $S \times S$. We have already noted that J is weakly decreasing in its second argument. The following lemma further characterizes J for the case in which the jurors use the same cutoff strategy.

Lemma 2 *J is continuous and weakly decreasing in its first argument. In addition,*

$$\lim_{s \downarrow \underline{S}} J(s, s) > 0 \quad \text{and} \quad \lim_{s \uparrow \overline{S}} J(s, s) < 0,$$

and thus $s^* = \inf\{s \in S \mid J(s, s) \leq 0\} \in S$. Finally, $J(s, s) = 0$ has at most one solution. □

The following theorem establishes existence of a symmetric, responsive cutoff equilibrium and uniqueness within the class of symmetric, responsive cutoff profiles. If the likelihood ratio is strictly decreasing, there are no non-cutoff symmetric, responsive equilibria.

Theorem 1 *There exists a symmetric, responsive cutoff equilibrium with cutoff given by $s^* = \inf\{s \in S \mid J(s, s) \leq 0\}$. It is unique within the class of symmetric, responsive cutoff profiles. If the likelihood ratio is locally strictly decreasing at s^*, then this equilibrium is unique a.e. within the class of all symmetric, responsive profiles.* □

Proof By Lemma 1, s^* defines a symmetric, responsive cutoff equilibrium if and only if $s^* \in S$, $J(s^*, s) \geq 0$ for all $s < s^*$, and $J(s^*, s) \leq 0$ for all $s > s^*$. The first condition, that $s^* \in S$, follows directly from Lemma 2. Take any $s < s^*$ and suppose $J(s^*, s) < 0$. Since J is continuous in its first argument, by Lemma 2, there is some $\epsilon > 0$ such that $s^* - \epsilon > s$ and $J(s^* - \epsilon, s) < 0$. Since J is weakly decreasing in its second argument, $J(s^* - \epsilon, s^* - \epsilon) < 0$, contradicting our definition of s^*. Therefore, $s < s^*$ implies $J(s^*, s) \geq 0$. Now take any $s > s^*$ and suppose $J(s^*, s) > 0$. Since J is continuous in its first argument, there is some $\epsilon > 0$ such that $s^* + \epsilon < s$ and $J(s^* + \epsilon, s) > 0$. Since J is weakly decreasing in its second argument, $J(s^* + \epsilon, s^* + \epsilon) > 0$. Since J is also weakly decreasing in its first argument, by Lemma 2, $J(\hat{s}, \hat{s}) > 0$ for all $\hat{s} < s^* + \epsilon$, contradicting the definition of s^*. Therefore, $s > s^*$ implies $J(s^*, s) \leq 0$, giving us the first part of the theorem.

To prove the second part of the theorem, consider any $s' \in S$ such that $J(s', s) \geq 0$ for all $s < s'$ and $J(s', s) \leq 0$ for all $s > s'$. If $s' < s^*$, take $\epsilon > 0$ such that $s^* - \epsilon > s'$. Then $J(s', s^* - \epsilon) \leq 0$. Since J is weakly decreasing in its first argument, by Lemma 2, $J(s^* - \epsilon, s^* - \epsilon) \leq 0$, contradicting our definition of s^*. If $s' > s^*$, take $\epsilon > 0$ such that $s^* + \epsilon < s'$. Then $J(s', s^* + \epsilon) \geq 0$. Since J is weakly decreasing in its first argument, $J(s^* + \epsilon, s^* + \epsilon) \geq 0$. Since J is weakly decreasing in its second argument as well, $J(s, s) \geq 0$ for all $s < s^* + \epsilon$. But $J(s, s) = 0$ has at most one solution, by Lemma 2, so $J(s, s) > 0$ for all $s < s^* + \epsilon$, contradicting our definition of s^*.

To prove the third part of the theorem, assume that the likelihood ratio is locally strictly decreasing at s^*. In any symmetric, responsive equilibrium, the jurors use the same best response strategy. By Lemma 1, this strategy is equivalent a.e. to a cutoff strategy, and we just proved uniqueness within the class of symmetric, responsive cutoff profiles. ■

A comparative statics result is immediate: since $J(s, s)$ is weakly decreasing in s, s^* is weakly decreasing in ρ. Intuitively, if the relative probability of guilt becomes higher, or if the relative cost of acquittal becomes higher, s^* tends downward. Each juror's probability of voting to convict tends upward accordingly.

It is easily checked that, as long as $\underline{S}, \overline{S} \in \Re$, there is a symmetric strategy profile that maximizes the *ex ante* payoff of the jurors among the class of symmetric strategy profiles: we identify a symmetric strategy profile $\sigma =$

$(\hat{\sigma}, \ldots, \hat{\sigma})$ with a point $\hat{\sigma}$ in the unit ball of $\mathcal{L}_\infty(S)$; endowing the unit ball with the weak topology, the subset of strategies $\hat{\sigma}\colon \Re \to [0,1]$ is compact, and the jurors' *ex ante* payoff is continuous in $\hat{\sigma}$; therefore, an optimum exists. Moreover, by (A4), the optimal symmetric strategy profile must be responsive. Though McLennan (1998) considers a model with finite types, the proof of his Theorem 2 uses only linearity of the jurors' *ex ante* payoff and translates directly to our model, with the conclusion that every optimal symmetric strategy profile is an equilibrium. By Theorem 1, there is only one such profile if the likelihood ratio is strictly decreasing, and it is the *unique* symmetric, responsive cutoff equilibrium.

4 Examples

We give three examples to illustrate the general model developed above. The first is a continuous analogue of the binary signal model of Feddersen and Pesendorfer (1998), in which our pure strategy cutoff equilibria can be interpreted as purifications of the mixed strategy equilibria of their model. More generally, any discrete signal model could be matched with a continuous analogue in a similar way. In the second example, the signals of the jurors are exponentially distributed, a particularly tractable functional form: the ratio of hazard rates is constant (a convenient property in later sections), and the probabilities of convicting an innocent and of acquitting a guilty defendant are positive and independent of the number of jurors under unanimity rule, anticipating our result on the asymptotic inefficiency of unanimity rule under certain conditions. In the third example, the signals of the jurors have a chi-square distribution. Here, the probabilities of convicting an innocent and of acquitting a guilty defendant converge to zero as the jury size increases, anticipating a result on the asymptotic probability of convicting an innocent.

4.1 The Binary Signal Model

To define the *binary signal model* in our framework, let $S = (0, 2)$,

$$f(s|I) = \begin{cases} p & \text{if } 0 < s \leq 1 \\ 1-p & \text{if } 1 < s < 2, \end{cases}$$

and

$$f(s|G) = \begin{cases} 1-p & \text{if } 0 < s \le 1 \\ p & \text{if } 1 < s < 2. \end{cases}$$

Obviously, the likelihood ratio is bounded over S. In order to satisfy (A3) and (A4), we impose $1/2 < p < 1$ and

$$\frac{p}{1-p} > \rho > \frac{1-p}{p}.$$

It follows that

$$J(s,s) = \begin{cases} \left(\frac{1-sp}{1-s+sp}\right)^{k-1} \left(\frac{p}{1-p}\right)^{n-k+1} - \rho & \text{if } 0 < s \le 1 \\ \left(\frac{1-p}{p}\right)^{k} \left(\frac{s-1-ps+2p}{1+ps-2p}\right)^{n-k} - \rho & \text{if } 1 < s < 2. \end{cases}$$

Note that J is weakly decreasing (strictly so if $1 < k < n$), and that it is discontinuous at $s = 1$. See Figure 1.

[Figure 1 about here.]

Recall that s^* is defined in the previous section as $\inf\{s \in S \mid J(s,s) \le 0\}$. Since

$$J(1,1) = \left(\frac{p}{1-p}\right)^{n-2k+2} - \rho$$
$$\lim_{s\downarrow 1} J(s,s) = \left(\frac{p}{1-p}\right)^{n-2k} - \rho,$$

we see that $J(1,1) \ge 0$ if and only if (1) $k < \frac{n}{2}+1$ or (2) both $k = \frac{n}{2}+1$ and $\rho \le 1$. And $\lim_{s\downarrow 1} J(s,s) \le 0$ if and only if (3) $k > \frac{n}{2}$ or (4) both $k = \frac{n}{2}$ and $\rho \ge 1$. We conclude that $s^* < 1$ if neither (1) nor (2) hold; $s^* > 1$ if neither (3) nor (4) hold; in those cases, s^* is given by the unique solution to $J(s,s) = 0$. In the remaining cases, $s^* = 1$.

In particular, we note that if a supermajority is required to convict, then $s^* < 1$ necessarily; if a majority is required and $\rho \ge 1$, again $s^* < 1$. That is, in equilibrium, jurors who receive some signals below one (indicating innocence) will vote to acquit, while those who receive other signals above one will vote to convict. Note also that $s^* = 1$ holds if and only if either both $k = \frac{n}{2} + 1$ and $\rho < 1$, or else both $k = \frac{n}{2}$ and $\rho \geq 1$. Thus, it is an equilibrium for jurors to "vote with their signals" only under majority rule (or close to it) and then only for restricted ρ's.

In the model of Feddersen and Pesendorfer (1998), jurors get one of only two possible signals: a signal that innocence is likely, which occurs with probability p if the defendant is innocent and with probability $1 - p$ if the defendant is guilty, and a signal that guilt is likely, which occurs with the same probabilities reversed. Our example replaces the innocence signal with a continuum of signals from 0 to 1, and the guilt signal with a continuum of signals from 1 to 2. Whereas jurors who receive the innocence signal in their model vote to acquit with some probability, say $a < 1/2$, and to convict with some probability $1 - a$, we partition $(0, 1)$ into two intervals $(0, a)$ and $(a, 1)$; jurors who receive signals in $(0, a)$ vote to acquit, and those who receive signals in $(a, 1)$ vote to convict. Thus, the cutoff equilibrium in our version of the binary signal model is a purification of the mixed strategy equilibrium in their model.

Theorem 1 guarantees existence of a symmetric, responsive cutoff equilibrium and uniqueness within that class; but because the likelihood ratio is not anywhere locally strictly decreasing in the continuous version of the binary signal model, the theorem does not guarantee uniqueness within the larger class of all symmetric, responsive profiles. Indeed, because jurors who receive signals between 0 and 1 are indifferent between voting to convict and voting to acquit, there is a continuum of symmetric, responsive non-cutoff equilibria: we could specify any subset of $(0, 1)$ with Lebesgue measure a and have jurors receiving signals therein vote to acquit, jurors receiving other signals vote to convict.

Before leaving the binary signal model, we calculate the hazard rates

$$\frac{f(s|I)}{1 - F(s|I)} = \begin{cases} \frac{p}{1-sp} & \text{if } 0 < s \leq 1 \\ \frac{1}{2-s} & \text{if } 1 < s < 2 \end{cases}$$

and

$$\frac{f(s|G)}{1 - F(s|G)} = \begin{cases} \frac{1-p}{1-s+sp} & \text{if } 0 < s \leq 1 \\ \frac{1}{2-s} & \text{if } 1 < s < 2. \end{cases}$$

Thus, the ratio of the hazard rate when innocent to the hazard rate when guilty increases from $p/(1-p)$ at $s=0$ to $(p/(1-p))^2$ at $s=1$, and then drops to one for $1<s<2$.

4.2 The Exponential Model

To define the *exponential model,* let $S=(0,\infty)$, $f(s|I)=\lambda e^{-\lambda s}$, and $f(s|G)=\gamma e^{-\gamma s}$. Again, the likelihood ratio is bounded. In order to satisfy (A3) and (A4), we need $\lambda>\gamma$ and $\lambda/\gamma>\rho$. It follows that

$$J(s,s) \quad = \quad \left(\frac{e^{-\lambda s}}{e^{-\gamma s}}\right)^{k-1}\left(\frac{1-e^{-\lambda s}}{1-e^{-\gamma s}}\right)^{n-k}\frac{\lambda e^{-\lambda s}}{\gamma e^{-\gamma s}}-\rho,$$

which is strictly decreasing and continuous on S. See Figure 2.

[Figure 2 about here.]

The unique symmetric equilibrium is found by solving $J(s^*,s^*)=0$; for the special case of unanimity rule, we readily obtain

$$s^* \quad = \quad \frac{1}{(\lambda-\gamma)n}\ln\left(\frac{\lambda}{\gamma\rho}\right).$$

The hazard rates when innocent and guilty are simply λ and γ, respectively, so the ratio of hazard rates is constant. (Other examples with constant ratios of hazard rates can be easily obtained from certain parameterizations of the Pareto and Weibull distributions.) Note that, under unanimity rule, the probability of convicting the defendant conditional on innocence is

$$(1-F(s^*|I))^n \quad = \quad \left(\frac{\rho\gamma}{\lambda}\right)^{\frac{\lambda}{\lambda-\gamma}},$$

and the probability of acquitting the defendant, conditional on guilt, is

$$1-(1-F(s^*|G))^n \quad = \quad 1-\left(\frac{\rho\gamma}{\lambda}\right)^{\frac{\gamma}{\lambda-\gamma}}.$$

Both probabilities are strictly positive and independent of the size of the jury. As a consequence, the probability of a mistake does not diminish as the size of the jury increases.

4.3 The Chi-Square Model

To define the *chi-square model,* let $S = (0, \infty)$ and define

$$f(s|I) = \frac{(1/2)^{\mu/2}}{\Gamma(\mu/2)} s^{(\mu/2)-1} e^{-s/2}$$

and

$$f(s|G) = \frac{(1/2)^{\nu/2}}{\Gamma(\nu/2)} s^{(\nu/2)-1} e^{-s/2}$$

where μ and ν are natural numbers and $\Gamma(\cdot)$ is the gamma function. In order to satisfy (A3) and (A4), we need $\nu > \mu$. Note that the likelihood ratio is unbounded and the ratio of hazard rates is strictly decreasing, two conditions important in the following sections.

One numerical example is $\mu = 2$ and $\nu = 4$. In this case

$$J(s,s) = \left(\frac{1}{(s/2)+1}\right)^{k-1} \left(\frac{1-e^{-s/2}}{1-((s/2)+1)e^{-s/2}}\right)^{n-k} \left(\frac{2}{s}\right) - \rho,$$

which is strictly decreasing and continuous on S. The unique symmetric equilibrium is found by solving $J(s^*, s^*) = 0$; for the special case of unanimity rule, we obtain s^* implicitly as the solution of the equation

$$2^n = (s^* + 2)^{n-1} s^* \rho.$$

The probability of convicting the defendant conditional on innocence is

$$(1 - F(s^*|I))^n = e^{-s^* n/2},$$

and the probability of acquitting the defendant, conditional on guilt, is

$$1 - (1 - F(s^*|G))^n = 1 - ((s^*/2) + 1)^n e^{s^* n/2}.$$

Using these expressions, we can establish the following proposition.

Proposition 2 *In the chi-square model with $\mu = 2$ and $\nu = 4$, the probabilities of convicting an innocent and acquitting a guilty defendant under unanimity rule converge to zero as the jury size increases.* □

Thus, we have a conclusion in stark contrast to that derived for the exponential model, where the probabilities of mistakes are invariant with respect to the size of the jury. As we will see in the next section, Proposition 2 also contrasts with the asymptotic properties of unanimity rule in the binary signal model. The key is that the likelihood ratio is bounded in the binary signal and exponential models, but not in the chi-square.

5 Unanimity Rule

In this section we investigate jury decision-making under unanimity rule. We provide three results. First, we give conditions under which the symmetric, responsive cutoff equilibrium is unique among the class of all responsive profiles, dropping the qualification of symmetry. Second, we identify conditions under which unanimity rule exhibits a bias in favor of convicting innocent defendants, regardless of the size of the jury. Third, we investigate the asymptotic properties of unanimity rule: if the likelihood ratio is bounded, so that the likelihood of innocence cannot be arbitrarily great, then the probability of a mistaken judgment is bounded strictly above zero; if the likelihood ratio is unbounded, then the probability of convicting an innocent goes to zero as n grows large. From the previous section, the former applies to the binary signal and exponential models, while the latter applies to the chi-square model. Define

$$H(s|I) = \frac{f(s|I)}{1 - F(s|I)} \quad \text{and} \quad H(s|G) = \frac{f(s|G)}{1 - F(s|G)},$$

the hazard rates when the defendant is innocent and when guilty.

Theorem 2 *If the ratio of hazard rates is strictly monotone and continuous, then the symmetric, responsive cutoff equilibrium under unanimity rule is unique within the class of all responsive cutoff profiles. If in addition the likelihood ratio is strictly decreasing, then this equilibrium is unique a.e. within the class of all responsive profiles.* □

Proof By Theorem 1, we know that there exists a unique symmetric, responsive cutoff equilibrium. Let $s^1, \ldots, s^n \in S$ be the cutoffs used by jurors $i = 1, \ldots, n$ in any responsive cutoff equilibrium. We will show that

$s^1 = \cdots = s^n$. Define

$$J^i(s^1, \ldots, s^n) \;=\; \left[\Pi_{j\neq i}\left(\frac{1-F(s^j|I)}{1-F(s^j|G)}\right)\right]\frac{f(s^i|I)}{f(s^i|G)} - \rho.$$

By Lemma 1, $J^i(s^1, \ldots, s^{i-1}, s, s^{i+1}, \ldots s^n)$ is greater than or equal to zero for all $s < s^i$ and less than or equal to zero for all $s > s^i$. Because the ratio of hazard rates is continuous, the likelihood ratio is as well, and so $J^i(s^1, \ldots, s^n)$ is continuous. Thus, $J^i(s^1, \ldots, s^n) = 0$ for all i. This implies that, for any two distinct jurors, i and j,

$$\left(\frac{1-F(s^i|I)}{1-F(s^i|G)}\right)\frac{f(s^j|I)}{f(s^j|G)} \;=\; \left(\frac{1-F(s^j|I)}{1-F(s^j|G)}\right)\frac{f(s^i|I)}{f(s^i|G)}$$

or, equivalently,

$$\frac{H(s^j|I)}{H(s^j|G)} \;=\; \frac{H(s^i|I)}{H(s^i|G)},$$

which, given strict monotonicity of the ratio of hazard rates, implies $s^i = s^j$. This establishes the uniqueness of the symmetric, responsive cutoff equilibrium within the class of all responsive cutoff profiles. If the likelihood ratio is strictly decreasing, all responsive equilibria are cutoff by Lemma 1, and uniqueness a.e. within the class of all responsive profiles follows. ■

Because the ratio of hazard rates is strictly decreasing and continuous in the chi-square model, Theorem 2 establishes that there are no asymmetric, responsive equilibria in that model. For the binary signal model, Theorem 2 suggests that asymmetric, responsive equilibria are possible under unanimity rule because the ratio of hazard rates is discontinuous at $s = 1$; moreover, it is constant for $1 < s < 2$. For the exponential model, the theorem suggests that asymmetric, responsive equilibria are possible, because the ratio of hazard rates is constant for every $s \in S$. In fact, it is easy to show that there is a continuum of asymmetric, responsive cutoff equilibria in the exponential model: any cutoffs $s_1, s_2, \ldots, s_n > 0$ for the jurors satisfying

$$s^1 + \cdots + s^n \;=\; \frac{1}{(\lambda-\gamma)}\ln\left(\frac{\lambda}{\gamma\rho}\right)$$

corresponds to a responsive cutoff equilibrium. (The symmetric equilibrium identified in the previous section is obtained by setting all cutoffs equal.) Note, however, that all these equilibria lead to the same probabilities of convicting an innocent and acquitting a guilty defendant, and, hence, they all lead to same expected payoff for jurors.

We turn now to a comparison between unanimity rule and other voting rules for an arbitrarily fixed jury size: we give a sufficient condition for unanimity rule to exhibit a bias in favor of convicting innocent defendants. We write s_k for the cutoff corresponding to the symmetric, responsive cutoff equilibrium when the number of votes needed to convict is k. Thus, the cutoff corresponding to unanimity rule is given by s_n, and the cutoff corresponding to simple majority rule (with an odd number of voters) is given by $s_{(n+1)/2}$. We write J_k to make explicit the dependence of J on the decision rule. We write $P_k(C|I)$ for the probability of conviction conditional on innocence and $P_k(A|G)$ for the probability of acquittal conditional on guilt under the symmetric, responsive cutoff equilibrium, when the number of votes needed to convict is k.

Theorem 3 *For all $k = 1, 2, \ldots, n-1$, if*

$$\lim_{s \uparrow s_n} \frac{H(s|I)}{H(s|G)} \leq \lim_{s \downarrow s_k} \frac{H(s|I)}{H(s|G)},$$

then

$$\frac{P_n(C|I)}{1 - P_n(A|G)} > \frac{P_k(C|I)}{1 - P_k(A|G)}. \qquad \square$$

Proof Note that

$$\frac{P_n(C|I)}{1 - P_n(A|G)} = \frac{(1 - F(s_n|I))^n}{(1 - F(s_n|G))^n}$$

and, for $k = 1, 2, \ldots, n-1$,

$$\frac{P_k(C|I)}{1 - P_k(A|G)} = \frac{\sum_{m=k}^{n} \binom{n}{m} (1 - F(s_k|I))^m (F(s_k|I))^{n-m}}{\sum_{m=k}^{n} \binom{n}{m} (1 - F(s_k|G))^m (F(s_k|G))^{n-m}}$$

$$= \frac{(1 - F(s_k|I))^k (F(s_k|I))^{n-k} \left[\binom{n}{k} + \sum_{m=k+1}^{n} \binom{n}{m} \left(\frac{1 - F(s_k|I)}{F(s_k|I)} \right)^{m-k} \right]}{(1 - F(s_k|G))^k (F(s_k|G))^{n-k} \left[\binom{n}{k} + \sum_{m=k+1}^{n} \binom{n}{m} \left(\frac{1 - F(s_k|G)}{F(s_k|G)} \right)^{m-k} \right]}.$$

By (4) of Lemma 0, on strict first order stochastic dominance,

$$\sum_{m=k+1}^{n} \binom{n}{m} \left(\frac{1-F(s_k|I)}{F(s_k|I)}\right)^{m-k} \quad < \quad \sum_{m=k+1}^{n} \binom{n}{m} \left(\frac{1-F(s_k|G)}{F(s_k|G)}\right)^{m-k}.$$

Hence, for $k = 1, 2, \ldots, n-1$,

$$\frac{P_k(C|I)}{1-P_k(A|G)} \quad < \quad \frac{(1-F(s_k|I))^k (F(s_k|I))^{n-k}}{(1-F(s_k|G))^k (F(s_k|G))^{n-k}},$$

so it remains only to be shown that

$$\left(\frac{1-F(s_n|I)}{1-F(s_n|G)}\right)^n \quad \geq \quad \left(\frac{1-F(s_k|I)}{1-F(s_k|G)}\right)^k \left(\frac{F(s_k|I)}{F(s_k|G)}\right)^{n-k}$$

for $k = 1, 2, \ldots, n-1$.

Take any $\epsilon > 0$ such that $s_n - \epsilon \in S$ and $s_k + \epsilon \in S$. By definition of s_n and s_k and because J is weakly decreasing in both arguments, $J_n(s_n - \epsilon, s_n - \epsilon) > 0 \geq J_k(s_k + \epsilon, s_k + \epsilon)$. Note that we can write

$$J_k(s,s) \quad = \quad \left(\frac{1-F(s|I)}{1-F(s|G)}\right)^k \left(\frac{F(s|I)}{F(s|G)}\right)^{n-k} \frac{H(s|I)}{H(s|G)} - \rho$$

for all $s \in S$, giving us

$$\begin{aligned} &\left(\frac{1-F(s_n-\epsilon|I)}{1-F(s_n-\epsilon|G)}\right)^n \frac{H(s_n-\epsilon|I)}{H(s_n-\epsilon|G)} \\ &\qquad > \left(\frac{1-F(s_k+\epsilon|I)}{1-F(s_k+\epsilon|G)}\right)^k \left(\frac{F(s_k+\epsilon|I)}{F(s_k+\epsilon|G)}\right)^{n-k} \frac{H(s_k|I)}{H(s_k|G)}. \end{aligned}$$

Taking limits and using continuity of $F(\cdot|I)$ and $F(\cdot|G)$, we have

$$\begin{aligned} &\left(\frac{1-F(s_n|I)}{1-F(s_n|G)}\right)^n \lim_{s \uparrow s_n} \frac{H(s|I)}{H(s|G)} \\ &\qquad \geq \left(\frac{1-F(s_k|I)}{1-F(s_k|G)}\right)^k \left(\frac{F(s_k|I)}{F(s_k|G)}\right)^{n-k} \lim_{s \downarrow s_k} \frac{H(s|I)}{H(s|G)}, \end{aligned}$$

and the assumption of the theorem delivers the desired inequality. ■

A direct implication of Theorem 3 is that, under the ratio of hazard rates condition, unanimity rule does not dominate any other rule in terms of mistake probabilities: $P_n(C|I) < P_k(C|I)$ implies $P_n(A|G) > P_k(A|G)$; and $P_n(A|G) < P_k(A|G)$ implies $P_n(C|I) > P_k(C|I)$. The theorem has an intuitive interpretation when $P(G) = P(I)$, in which case it follows that

$$\frac{P_n(C|I)P(I)}{P_n(C|G)P(G) + P_n(C|I)P(I)} > \frac{P_k(C|I)P(I)}{P_k(C|G)P(G) + P_k(C|I)P(I)}.$$

That is, the probability that the defendant is innocent, conditional on conviction, is higher under unanimity rule than when k votes are required to convict.

To give a more transparent sufficient condition for the result of Theorem 3, we use the following lemma, which establishes that s_k is weakly decreasing in k. That is, as the number of votes required to convict the defendant increases, jurors become more willing to vote for convicting.

Lemma 3 *If $n \geq k' > k \geq 1$, then $s_{k'} \leq s_k$.* □

We can now state a corollary of Theorem 3. The proof, given Theorem 3 and Lemma 3, is straightforward.

Corollary 1 *If $H(s|I)/H(s|G)$ is weakly increasing on an open interval including $[s_n, s_k]$, then*

$$\frac{P_n(C|I)}{1 - P_n(A|G)} > \frac{P_k(C|I)}{1 - P_k(A|G)},$$

for all $k = 1, 2, \ldots, n-1$. □

Applied to the binary signal model, because the ratio of hazard rates is strictly increasing from 0 to 1, we know that the conclusion of the corollary holds if $s_k < 1$. In particular, it holds if $k > \frac{n}{2} + 1$ or if both $k = \frac{n}{2} + 1$ and $\rho \geq 1$. (See Feddersen and Pesendorfer's (1998) Table 1 for numerical values when $n = 12$.) Applied to the exponential model, because the ratio of hazard rates is constant, unanimity rule exhibits a bias toward convicting the innocent compared to any other voting rule. Note, however, that the bias in Theorem 3 is expressed in terms of the *ratio* of the probability of convicting an innocent to the probability of convicting a guilty defendant, and not simply in terms of the probability of convicting an innocent. In the exponential model, if $n = 3, \rho = 1, \lambda = 1/2$, and $\gamma = 1/3$, then the probability of convicting an innocent under unanimity rule is (approximately) .2963 while under majority rule it is .3061. In agreement with Theorem 3, the ratio of the probability of convicting an innocent to the probability of convicting a guilty defendant is 2/3 under unanimity rule and .5889 under majority rule.

Theorem 3 cannot be applied to the chi-square model because the ratio of hazard rates is decreasing. Indeed, if $n = 3, \rho = 1, \mu = 2$, and $\nu = 4$, the probabilities of convicting an innocent and acquitting a guilty defendant under unanimity rule are .2474 and .2212, while the same probabilities under majority rule are .2674 and .2083. Hence, majority rule is biased towards convicting an innocent with respect to unanimity rule in the sense of a higher probability of convicting an innocent and in the sense of a higher ratio of that probability to the probability of convicting a guilty defendant. Moreover, the expected payoff for jurors is higher under unanimity rule than under majority rule.

Finally, we turn to the asymptotic properties of unanimity rule. To anticipate the notation of the asymptotic results on other voting rules in the next section, we write s_1^n for the cutoff corresponding to the symmetric, responsive cutoff equilibrium under unanimity rule when the jury size is n. (The subscript indicates the guilty vote, as a fraction of all jurors, needed to convict.) Similarly, we write J_1^n for the function J and $P_1^n(C|I)$ and $P_1^n(A|G)$ for the probabilities of convicting an innocent and acquitting a guilty defendant. The following lemma establishes that, under unanimity rule, the cutoff corresponding to the symmetric, responsive cutoff equilibrium converges to the lower bound of the support of the signals as the number of jurors increases. Thus, the probability that a given juror votes to acquit goes to zero.

Lemma 4 $\lim_{n\to\infty} s_1^n = \underline{S}$. □

An implication of Theorem 4, next, is that the probability of a mistaken judgment under unanimity rule is bounded strictly above zero as the number of jurors increases, if the likelihood ratio is bounded. Thus, we infer the asymptotic inefficiency of unanimity rule in the binary signal and exponential models. If the the likelihood ratio is unbounded, as in the chi-square model, we have seen in Proposition 2 that unanimity rule may be asymptotically efficient. In fact, we prove that the probability of convicting an innocent defendant must then go to zero as the number of jurors increases.

Theorem 4 *If the likelihood ratio is bounded, then*

$$0 \;<\; \lim_{n\to\infty} \frac{P_1^n(C|I)}{1 - P_1^n(A|G)} \;<\; \infty.$$

If the likelihood ratio is unbounded, then

$$\lim_{n\to\infty} \frac{P_1^n(C|I)}{1 - P_1^n(A|G)} \quad = \quad 0. \qquad \square$$

PROOF By (A1), the likelihood ratio has at most a finite number of discontinuity points, so by Lemma 4 there exists m such that, for all $n > m$, s_1^n is a continuity point of J_1^n. Hence, $J_1^n(s_1^n, s_1^n) = 0$ for all such n. Thus,

$$\frac{P_1^n(C|I)}{1 - P_1^n(A|G)} = \left(\frac{1 - F(s_1^n|I)}{1 - F(s_1^n|G)}\right)^n = \rho \frac{H(s_1^n|G)}{H(s_1^n|I)}$$

for all $n > m$, where the second inequality is just rewriting $J_1^n(s_1^n, s_1^n) = 0$. If the likelihood ratio is bounded above by some κ, then

$$\lim_{s\to\underline{S}} \frac{H(s|I)}{H(s|G)} \leq \kappa,$$

where we use $\lim_{s\to\underline{S}}(1 - F(s|I))/(1 - F(s|G)) = 1$. Since $s_1^n \to \underline{S}$ by Lemma 4, we have

$$\lim_{n\to\infty} \frac{P_1^n(C|I)}{1 - P_1^n(A|G)} = \rho \lim_{s_1^n\to\underline{S}} \frac{H(s_1^n|G)}{H(s_1^n|I)} \geq \frac{\rho}{\kappa}.$$

The first limit above is clearly finite and positive. If the likelihood ratio is unbounded above, then $\lim_{s\to\underline{S}} H(s|I)/H(s|G) = \infty$. Since $s_1^n \to \underline{S}$ by Lemma 4,

$$\lim_{n\to\infty} \frac{P_1^n(C|I)}{1 - P_1^n(A|G)} = \rho \lim_{s_1^n\to\underline{S}} \frac{H(s_1^n|G)}{H(s_1^n|I)} = 0,$$

as desired. ■

6 A Jury Theorem

In this section, we investigate the asymptotic properties of jury decision rules other than unanimity as the jury size increases: the probability of a mistaken judgment goes to zero for every non-unanimous voting rule. Rather than specify the number of votes needed to convict, we will here define a rule by the fraction, say α, of votes needed. Given n, the decision rule requiring k votes to convict would be represented by $\alpha = k/n$. For ease of exposition, we only consider combinations of α and n such that αn is an integer. We write s_α^n for the cutoff corresponding to the symmetric, responsive cutoff equilibrium when the α rule is used and the number of jurors is n. Similarly, we write J_α^n for the function J and $P_\alpha^n(C|I)$ and $P_\alpha^n(A|G)$ for the probabilities of convicting an innocent and acquitting a guilty defendant. We first verify that the ratio of hazard rates is effectively bounded for all non-unanimous rules.

Lemma 5 *If* $0 < \alpha < 1$ *then*

$$\overline{H} \;=\; \limsup_{n\to\infty} \frac{H(s_\alpha^n|I)}{H(s_\alpha^n|G)},$$

is finite. □

We now state and prove the main result of this section.

Theorem 5 *For all* $0 < \alpha < 1$,

$$\lim_{n\to\infty} P_\alpha^n(C|I) \;=\; \lim_{n\to\infty} P_\alpha^n(A|G) \;=\; 0.$$

□

Proof Note that we can write

$$\begin{aligned} J_\alpha^n(\bar{s}, s) &= \left(\frac{1-F(\bar{s}|I)}{1-F(\bar{s}|G)}\right)^{\alpha n-1} \left(\frac{F(\bar{s}|I)}{F(\bar{s}|G)}\right)^{n-\alpha n} \frac{f(s|I)}{f(s|G)} - \rho \\ &= [L_\alpha(\bar{s})]^n \left(\frac{f(s|I)}{f(s|G)}\right)\left(\frac{1-F(\bar{s}|G)}{1-F(\bar{s}|I)}\right) - \rho \\ &= [L_\alpha(s)]^n \frac{H(s|I)}{H(s|G)} - \rho, \end{aligned}$$

where we define

$$L_\alpha(s) \quad = \quad \left(\frac{1-F(s|I)}{1-F(s|G)}\right)^\alpha \left(\frac{F(s|I)}{F(s|G)}\right)^{1-\alpha}$$

for all $s \in S$. From Theorem 1, for each n there is a unique symmetric, responsive cutoff equilibrium characterized by the cutoff $s_\alpha^n = \inf\{s \in S \mid J_\alpha^n(s,s) \leq 0\}$.

We claim that $L_\alpha(s_\alpha^n) \to 1$. If not, we can extract a subsequence with limsup greater than one or liminf less than one. Without loss of generality, we suppose this is true of $\{s_\alpha^n\}$ itself. In the first case, we can take m high enough that $[L_\alpha(s_\alpha^m)]^m > \rho$. Using continuity of L_α, we can then take $s > s_\alpha^m$ close enough to s_α^m that $[L_\alpha(s)]^m > \rho$. But then

$$J_\alpha^m(s,s) \geq [L_\alpha(s)]^m - \rho > \rho - \rho = 0,$$

contradicting the definition of s_α^m. In the second case, by Lemma 5

$$\overline{H} \quad = \quad \limsup_{n\to\infty} \frac{H(s_\alpha^n|I)}{H(s_\alpha^n|G)}$$

is finite. Take m high enough that $[L_\alpha(s_\alpha^m)]^m \overline{H} < \rho$. Using continuity of L_α, we can take $s < s_\alpha^m$ close enough to s_α^m so that $[L_\alpha(s)]^m \overline{H} < \rho$. But then

$$J_\alpha^m(s,s) \leq [L_\alpha(s)]^m \overline{H} - \rho < \rho - \rho = 0,$$

contradicting the definition of s_α^m.

We now claim that $L_\alpha(s) = 1$ implies $1 - F(s|G) > \alpha > 1 - F(s|I)$. We use the facts that $x^\alpha(1-x)^{1-\alpha}$ is single-peaked at $x = \alpha$ and, by strict first order stochastic dominance, $1 - F(s|I) < 1 - F(s|G)$ for all $s \in S$. If $\alpha \leq 1 - F(s|I)$ then $\alpha \leq 1 - F(s|I) < 1 - F(s|G)$ and, by single-peakedness,

$$(1-F(s|I))^\alpha (F(s|I))^{1-\alpha} \quad > \quad (1-F(s|G))^\alpha (F(s|G))^{1-\alpha},$$

or equivalently $L_\alpha(s) > 1$, a contradiction. Similarly, if $1 - F(s|G) \leq \alpha$, then $1 - F(s|I) < 1 - F(s|G) \leq \alpha$ and, by single-peakedness, $L_\alpha(s) < 1$, a contradiction establishing the claim. Since L_α is decreasing, continuous, and

$$\begin{array}{rcllll} \lim_{s\to\underline{S}} L_\alpha(s) & = & \lim_{s\to\underline{S}} \left(\frac{f(s|I)}{f(s|G)}\right)^{1-\alpha} & > & 1 \\ \lim_{s\to\overline{S}} L_\alpha(s) & = & \lim_{s\to\overline{S}} \left(\frac{f(s|I)}{f(s|G)}\right)^{\alpha} & < & 1 \end{array}$$

(using L'Hôpital's rule and (3) of Lemma 0), the set $L_\alpha^{-1}(1)$ is a non-empty closed interval, $[s', s'']$, with $\underline{S} < s' \leq s'' < \overline{S}$. By continuity of the distribution functions, we can take $\delta > 0$ such that $1 - F(s|G) > \alpha > 1 - F(s|I)$ for all $s \in [s' - \delta, s'' + \delta]$. Since $L_\alpha(s_\alpha^n) \to 1$, there exists l such that, for all $m > l$, $s_\alpha^m \in [s' - \delta, s'' + \delta]$.

The last part of the proof is a straightforward application of the law of large numbers. To prove $P_\alpha^n(A|G) \to 0$, define the probability space $S^\infty = S \times S \times \cdots$ with probability measure P, the product measure generated by μ_G. Define the sequence $X_1, X_2, \ldots$ of i.i.d. random variables satisfying

$$X_i = \begin{cases} 1 & \text{if } s_i \geq s'' + \delta \\ 0 & \text{else,} \end{cases}$$

where s_i is the ith component of $(s_1, s_2, \ldots) \in S^\infty$. By the strong law of large numbers, $\frac{1}{n}\sum_{i=1}^n X_i$ converges almost surely to $1 - F(s'' + \delta|G)$ as n goes to infinity. In particular, it converges in probability:

$$P\left(1 - F(s'' + \delta|G) - \frac{1}{n}\sum_{i=1}^n X_i > \epsilon\right) \to 0$$

for all $\epsilon > 0$. Define the sequence $Y_1, Y_2, \ldots$ of random variables as

$$Y_n = \frac{1}{n}\#\{i \leq n \mid s_i \geq s_\alpha^n\},$$

and note that, for $m > l$, $Y_n \geq \frac{1}{n}\sum_{i=1}^n X_i$. Hence,

$$P(1 - F(s'' + \delta|G) - Y_n > \epsilon) \to 0,$$

or equivalently,

$$P(Y_n < 1 - F(s'' + \delta|G) - \epsilon) \to 0$$

for all $\epsilon > 0$. Since $1 - F(s'' + \delta|G) > \alpha$, we can set $\epsilon = 1 - F(s'' + \delta|G) - \alpha$, yielding $P(Y_n < \alpha) \to 0$. That is, the probability that the fraction of jurors voting to convict a guilty defendant is smaller than α goes to zero as the size of the jury goes to infinity. Therefore, $P_\alpha^n(A|G) \to 0$. The proof that $P_\alpha^n(C|I) \to 0$ is analogous. ∎

Appendix

Proposition 1 *Given σ_{-i}, the ex ante payoff to juror i from σ_i is an affine transformation of*

$$\left(\int \sigma_i(s)\,\mu_I(ds)\right) u(C|I)P_{\sigma_{-i}}(piv|I)P(I) - \left(\int \sigma_i(s)\,\mu_G(ds)\right) u(A|G)P_{\sigma_{-i}}(piv|G)P(G).$$

Proof Let N denote the set of jurors. Note that

$$\begin{aligned} P_\sigma(C|I) &= \left(\int \sigma_i(s)\,\mu_I(ds)\right) \sum_{\substack{M\subseteq N\\ |M|\geq k-1\\ i\notin M}} \left[\Pi_{j\in M}\left(\int \sigma_j(s)\,\mu_I(ds)\right)\cdot \right.\\ &\quad \left. \Pi_{\substack{j\notin M\\ j\neq i}}\left(\int [1-\sigma_j(s)]\,\mu_I(ds)\right)\right]\\ &+ \left(\int [1-\sigma_i(s)]\,\mu_I(ds)\right) \sum_{\substack{M\subseteq N\\ |M|\geq k\\ i\notin M}} \left[\Pi_{j\in M}\left(\int \sigma_j(s)\,\mu_I(ds)\right)\cdot \right.\\ &\quad \left. \Pi_{\substack{j\notin M\\ j\neq i}}\left(\int [1-\sigma_j(s)]\,\mu_I(ds)\right)\right], \end{aligned}$$

and

$$\begin{aligned} P_\sigma(A|G) &= \left(\int \sigma_i(s)\,\mu_G(ds)\right) \sum_{\substack{M\subseteq N\\ |M|<k-1\\ i\notin M}} \left[\Pi_{j\in M}\left(\int \sigma_j(s)\,\mu_G(ds)\right)\cdot \right.\\ &\quad \left. \Pi_{\substack{j\notin M\\ j\neq i}}\left(\int [1-\sigma_j(s)]\,\mu_G(ds)\right)\right]\\ &+ \left(\int [1-\sigma_i(s)]\,\mu_G(ds)\right) \sum_{\substack{M\subseteq N\\ |M|<k\\ i\notin M}} \left[\Pi_{j\in M}\left(\int \sigma_j(s)\,\mu_G(ds)\right)\cdot \right.\\ &\quad \left. \Pi_{\substack{j\notin M\\ j\neq i}}\left(\int [1-\sigma_j(s)]\,\mu_G(ds)\right)\right]. \end{aligned}$$

Inserting these expressions into $u(C|I)P_\sigma(C|I)P(I) + u(A|G)P_\sigma(A|G)P(G)$ and simplifying, we get

$$\left(\int \sigma_i(s)\,\mu_I(ds)\right) u(C|I) \sum_{\substack{M\subseteq N \\ |M|=k-1 \\ i\notin M}} \left[\Pi_{j\in M}\left(\int \sigma_j(s)\,\mu_I(ds)\right)\cdot \right.$$
$$\left. \Pi_{\substack{j\notin M \\ j\neq i}}\left(\int [1-\sigma_j(s)]\,\mu_I(ds)\right)\right] P(I)$$
$$-\left(\int \sigma_i(s)\,\mu_G(ds)\right) u(A|G) \sum_{\substack{M\subseteq N \\ |M|=k-1 \\ i\notin M}} \left[\Pi_{j\in M}\left(\int \sigma_j(s)\,\mu_G(ds)\right)\cdot \right.$$
$$\left. \Pi_{\substack{j\notin M \\ j\neq i}}\left(\int [1-\sigma_j(s)]\,\mu_G(ds)\right)\right] P(G) + \text{ constant},$$

where the last term is independent of σ_i. ■

Lemma 0

(1) $$\frac{1-F(s|I)}{1-F(s|G)} \leq \frac{f(s|I)}{f(s|G)} \leq \frac{F(s|I)}{F(s|G)} \textit{ for all } s \in S.$$

(2) $$\frac{1-F(s|I)}{1-F(s|G)} \textit{ and } \frac{F(s|I)}{F(s|G)} \textit{ are weakly decreasing.}$$

(3) $$\lim_{s\downarrow \underline{S}} \frac{f(s|I)}{f(s|G)} > 1 \textit{ and } \lim_{s\uparrow \overline{S}} \frac{f(s|I)}{f(s|G)} < 1.$$

(4) $F(s|I) > F(s|G)$ *for all* $s \in S$.

(5) *If* $\frac{f(\tilde{s}|I)}{f(\tilde{s}|G)} > \lim_{s\uparrow \overline{S}} \frac{f(s|I)}{f(s|G)}$ *then* $\frac{1-F(\hat{s}|I)}{1-F(\hat{s}|G)} > \frac{1-F(\tilde{s}|I)}{1-F(\tilde{s}|G)}$ *for all* $\tilde{s} \in S$ *and all* $\hat{s} < \tilde{s}$.

(6) *If* $\frac{f(\tilde{s}|I)}{f(\tilde{s}|G)} < \lim_{s\downarrow \underline{S}} \frac{f(s|I)}{f(s|G)}$ *then* $\frac{F(\hat{s}|I)}{F(\hat{s}|G)} < \frac{F(\tilde{s}|I)}{F(\tilde{s}|G)}$ *for all* $\tilde{s} \in S$ *and all* $\hat{s} > \tilde{s}$.

Proof Results (1) and (2) follow from (A1)–(A3) and are well-known. Result (3) follows easily from (A3) and (A4). Result (4), stated with weak inequality (that is, first order stochastic dominance) is a well-known implication of (A3). Strict inequality follows from result (3) above. If (5) fails for $\tilde{s}$, then, by (2) above,

$$\frac{1-F(\tilde{s}|I)}{1-F(\tilde{s}|G)} = \frac{1-F(s|I)}{1-F(s|G)} \text{ for all } s \in [\hat{s}, \tilde{s}]$$

for some $\hat{s} < \overline{s}$. Consequently,

$$D\left(\frac{1-F(s|I)}{1-F(s|G)}\right) = 0,$$

or equivalently

$$\frac{1-F(s|I)}{1-F(s|G)} = \frac{f(s|I)}{f(s|G)}, \tag{7}$$

for all $s \in [\hat{s}, \tilde{s}] \cap S^d$. Note that

$$\frac{f(\tilde{s}|I)}{f(\tilde{s}|G)} > \lim_{s \uparrow \overline{S}} \frac{f(s|I)}{f(s|G)}$$

implies that there is a $s' > \tilde{s}$ such that

$$\frac{f(\tilde{s}|I)}{f(\tilde{s}|G)} > \frac{f(x|I)}{f(x|G)}$$

for all $x \geq s'$. Taking any $s \in (\hat{s}, \tilde{s})$ and using (7),

$$\begin{aligned} 1-F(s|I) &= \int_s^{\overline{S}} \frac{f(s|I)}{f(s|G)} f(x|G)\,dx \\ &= \int_s^{s'} \frac{f(s|I)}{f(s|G)} f(x|G)\,dx + \int_{s'}^{\overline{S}} \frac{f(s|I)}{f(s|G)} f(x|G)\,dx \\ &> \int_s^{s'} \frac{f(x|I)}{f(x|G)} f(x|G)\,dx + \int_{s'}^{\overline{S}} \frac{f(x|I)}{f(x|G)} f(x|G)\,dx \\ &= 1-F(s|I), \end{aligned}$$

where the inequality follows from (A3) and our choice of s'. But this is a contradiction, establishing (5). The proof of (6) is analogous. ■

Lemma 1 *Given responsive strategies σ_{-i} for jurors other than i, a strategy σ_i is a best response for i if and only if it satisfies the following a.e.:*

$$\sigma_i(s) \quad = \quad \begin{cases} 1 & \text{if } J(\sigma_{-i}, s) < 0 \\ 0 & \text{if } J(\sigma_{-i}, s) > 0. \end{cases} \tag{8}$$

If the likelihood ratio is locally strictly decreasing at $\inf\{s \in S \mid J(\sigma_{-i}, s) \leq 0\}$, *$\sigma_i$ is a best response for i if and only if it is equivalent a.e. to the following cutoff strategy $\tilde{\sigma}_i$:*

$$\tilde{\sigma}_i(s) \quad = \quad \begin{cases} 1 & \text{if } J(\sigma_{-i}, s) \leq 0 \\ 0 & \text{else.} \end{cases}$$

□

Proof Suppose σ_i satisfies (8). Take any strategy σ_i', and define the sets

$$\begin{aligned} V &= \{s \in S \mid J(\sigma_{-i}, s) < 0 \text{ and } \sigma_i'(s) < 1\} \\ W &= \{s \in S \mid J(\sigma_{-i}, s) > 0 \text{ and } \sigma_i'(s) > 0\}. \end{aligned}$$

Note that $\sigma_i(s) = 1$ for all $s \in V$ and $\sigma_i(s) = 0$ for all $s \in W$. Thus, using Proposition 1, the payoff from σ_i to juror i exceeds the payoff from σ_i' by

$$\begin{aligned} &\int_V (1 - \sigma_i'(s)) \big[u(C|I)P_{\sigma_{-i}}(piv|I)P(I)f(s|I) \\ &\qquad\qquad -u(A|G)P_{\sigma_{-i}}(piv|G)P(G)f(s|G)\big] \, ds \\ &- \int_W \sigma_i'(s) \big[u(C|I)P(I)P_{\sigma_{-i}}(piv|I)f(s|I) \\ &\qquad\qquad -u(A|G)P(G)P_{\sigma_{-i}}(piv|G)f(s|G)\big] \, ds. \end{aligned}$$

By construction, $s \in V$ implies $J(\sigma_{-i}, s) < 0$, which implies that the integrand of the first integral is positive; $s \in W$ implies $J(\sigma_{-i}, s) > 0$, which implies that the integrand of the second integral is negative. Since σ_i' violates (8) if and only if $V \cup W$ has positive measure, any strategy satisfying (8) is a best response and any strategy violating (8) is not.

If the likelihood ratio is locally strictly decreasing at $\inf\{s \in S \mid J(\sigma_{-i}, s) \leq 0\}$, $J(\sigma_{-i}, s) = 0$ has at most one solution, and hence (8) implies σ_i is equivalent a.e. to $\tilde{\sigma}_i$, a cutoff strategy. ■

Lemma 2 *J is continuous and weakly decreasing in its first argument. In addition,*

$$\lim_{s\downarrow\underline{S}} J(s,s) > 0 \quad \textit{and} \quad \lim_{s\uparrow\overline{S}} J(s,s) < 0,$$

and thus $s^* = \inf\{s \in S \mid J(s,s) \leq 0\} \in S$. Finally, $J(s,s) = 0$ has at most one solution. □

PROOF Continuity of J in its first argument is immediate, and weak monotonicity follows from (A3) and (2) of Lemma 0. Note that

$$\begin{aligned}
\lim_{s\downarrow\underline{S}} J(s,s) &= \lim_{s\downarrow\underline{S}} \left(\frac{F(s|I)}{F(s|G)}\right)^{n-k} \lim_{s\downarrow\underline{S}} \frac{f(s|I)}{f(s|G)} - \rho \\
&= \left(\lim_{s\downarrow\underline{S}} \frac{f(s|I)}{f(s|G)}\right)^{n-k+1} - \rho \\
&> 0,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{s\uparrow\overline{S}} J(s,s) &= \lim_{s\uparrow\overline{S}} \left(\frac{1-F(s|I)}{1-F(s|G)}\right)^{k-1} \lim_{s\uparrow\overline{S}} \frac{f(s|I)}{f(s|G)} - \rho \\
&= \left(\lim_{s\uparrow\overline{S}} \frac{f(s|I)}{f(s|G)}\right)^{k} - \rho \\
&< 0,
\end{aligned}$$

where we make use of L'Hôpital's rule, (A4), and (3) of Lemma 0. This proves the next part of the lemma. By (A1), $J(s,s)$ has at most a finite number of discontinuity points. Therefore, since $\lim_{s\downarrow\underline{S}} J(s,s) > 0$, we can find $s > \underline{S}$ close enough to $\underline{S}$ so that $J(s,s) > 0$. Thus, $s^* > \underline{S}$. A similar argument shows that $s^* < \overline{S}$. Thus, $s^* \in S$.

For the last part, take any signal $\overline{s}$ such that $J(\overline{s},\overline{s}) = 0$. We claim that, for all $s' > \overline{s}$, $J(s',s') < J(\overline{s},\overline{s}) = 0$. By the definition of J and (2) of Lemma 0, if

$$\frac{f(s'|I)}{f(s'|G)} < \frac{f(\overline{s}|I)}{f(\overline{s}|G)},$$

we are done. Thus, by (A3), we suppose the two likelihood ratios are equal. Note that, by (A4), either

$$\frac{f(\overline{s}|I)}{f(\overline{s}|G)} \quad < \quad \lim_{s \downarrow \underline{S}} \frac{f(s|I)}{f(s|G)}$$

or

$$\frac{f(s'|I)}{f(s'|G)} = \frac{f(\overline{s}|I)}{f(\overline{s}|G)} \quad > \quad \lim_{s \uparrow \overline{S}} \frac{f(s|I)}{f(s|G)}.$$

If the first inequality holds, then

$$\frac{F(s'|I)}{F(s'|G)} \quad < \quad \frac{F(\overline{s}|I)}{F(\overline{s}|G)}$$

by (6) of Lemma 0. If the second holds, then

$$\frac{1-F(s'|I)}{1-F(s'|G)} \quad < \quad \frac{1-F(\overline{s}|I)}{1-F(\overline{s}|G)}.$$

by (5) of Lemma 0.

We look at three cases. If $1 < k < n$, then by the definition of J and the preceding discussion, we are done. If $k = n$, then

$$J(s,s) \quad = \quad \left(\frac{1-F(s|I)}{1-F(s|I)}\right)^{n-1} \frac{f(s|I)}{f(s|G)} - \rho.$$

If

$$\frac{f(\overline{s}|I)}{f(\overline{s}|G)} \quad > \quad \lim_{s \uparrow \overline{S}} \frac{f(s|I)}{f(s|G)},$$

then by (5) of Lemma 0 we are done. Otherwise, we have

$$f(s|I) \quad = \quad f(s|G) \cdot \lim_{s \uparrow \overline{S}} \frac{f(s|I)}{f(s|G)}$$

for all $s \geq \overline{s}$. Then, after integrating and rearranging terms,

$$\frac{1-F(\overline{s}|I)}{1-F(\overline{s}|G)} \quad = \quad \lim_{s \uparrow \overline{S}} \frac{f(s|I)}{f(s|G)}.$$

Using $J(\overline{s}, \overline{s}) = 0$, we get

$$\rho \quad = \quad \left(\lim_{s \uparrow \overline{S}} \frac{f(s|I)}{f(s|G)} \right)^n .$$

But

$$\lim_{s \uparrow \overline{S}} \frac{f(s|I)}{f(s|G)} \quad < \quad 1,$$

by (3) of Lemma 0, so

$$\lim_{s \uparrow \overline{S}} \frac{f(s|I)}{f(s|G)} \quad > \quad \rho,$$

contradicting (A4). The case $k = 1$ is analogous. This establishes the claim that $s' > \overline{s}$ implies $J(s', s') < 0$. Therefore, $J(s, s) = 0$ has at most one solution. ∎

Proposition 2 *In the chi-square model with $\mu = 2$ and $\nu = 4$, the probabilities of convicting an innocent and acquitting a guilty defendant under unanimity rule converge to zero as the jury size increases.* □

PROOF Let y_n be the probability of convicting an innocent under unanimity rule when the jury size is n. Using the equations given in the text for y_n and s^* we obtain that y_n must solve

$$n^n \quad = \quad (n - \log y_n)^{n-1}(-\log y_n)\rho.$$

After some manipulations we obtain

$$\left(1 + \frac{-\log y_n}{n}\right)^n \quad = \quad \frac{1}{\rho} - \frac{n}{\rho \log y_n}.$$

Suppose some subsequence of y_n converges to some number $1 \geq \kappa > 0$. Then, because $(1 + (t/n))^n \to e^t$ as $n \to \infty$, the lefthand side converges to $1/\kappa$. Because $-\log y_n/n \to 0$, the righthand side converges to ∞, a contradiction. Hence, y_n converges to zero.

Now let x_n be the probability of acquitting a guilty defendant under unanimity rule when the jury size is n. Using the equations given in the text for x_n and y_n we obtain

$$x_n \;=\; 1 - \left(1 - \frac{\log y_n}{n}\right)^n y_n.$$

which converges to zero since, given a sequence $\{t_n\}$ that diverges to infinity, $(1 + t_n/n)^n e^{-t_n}$ converges to 1 if t_n/n converges to zero. (The proof of this fact is a slight variation of a standard exercise in mathematical analysis. In particular, to prove that $\liminf_{n\to\infty}(1 + t_n/n)^n e^{-t_n} = 1$, we can use the binomial theorem to show that the difference between e^{t_n} and the first $\lfloor 3t_n \rfloor -$ 1 terms in $(1 + t_n/n)^n$ converges to a number smaller than $3t_n^{\lfloor 3t_n \rfloor}/2(\lfloor 3t_n \rfloor)!$, which converges to zero by an application of Stirling's formula.) ■

Lemma 3 *If* $n \geq k' > k \geq 1$, *then* $s_{k'} \leq s_k$. □

Proof Recall that

$$J_k(s,s) \;=\; \left(\frac{1 - F(s \mid I)}{1 - F(s \mid G)}\right)^{k-1} \left(\frac{F(s \mid I)}{F(s \mid G)}\right)^{n-k} \frac{f(s \mid I)}{f(s \mid G)} - \rho.$$

By (2) of Lemma 0, given arbitrary $s \in S$,

$$\frac{1 - F(s \mid I)}{1 - F(s \mid G)} \;\leq\; \frac{F(s \mid I)}{F(s \mid G)},$$

which implies $J_{k'}(s,s) \leq J_k(s,s)$ for $k' > k$. This implies

$$\{s \in S \mid J_k(s,s) \leq 0\} \;\subseteq\; \{s \in S \mid J_{k'}(s,s) \leq 0\},$$

from which we conclude $s_{k'} \leq s_k$. ■

Lemma 4 $s_1^n \to \underline{S}$ □

Proof If not, we can extract a subsequence that converges to a limit larger than $\underline{S}$. Without loss of generality, suppose this is true of $\{s_1^n\}$ itself, so that $s_1^n \to \tilde{s} > \underline{S}$. Note that

$$\lim_{s_1^n \to \tilde{s}} \frac{1 - F(s_1^n|I)}{1 - F(s_1^n|G)} \;=\; \gamma \;<\; 1$$

by (4) of Lemma 0, or by (3) of Lemma 0 if $\tilde{s} = \overline{S}$. By continuity, we can pick $\hat{s} \in (\underline{S}, \tilde{s})$ such that

$$\frac{1 - F(\hat{s}|I)}{1 - F(\hat{s}|G)} \;\leq\; \frac{\gamma + 1}{2} \;<\; 1.$$

Then

$$J_1^n(\hat{s}, \hat{s}) \;\;\leq\;\; \left(\frac{\gamma + 1}{2}\right)^{n-1} \frac{f(\hat{s}|I)}{f(\hat{s}|G)} - \rho,$$

which is less than zero for high enough n. Because $s_1^n \to \tilde{s} > \hat{s}$, we can pick n high enough that $J_1^n(\hat{s}, \hat{s}) < 0$ and $s_1^n > \hat{s}$, a contradiction. ■

Lemma 5 *If* $0 < \alpha < 1$, *then*

$$\overline{H} \;\;=\;\; \limsup_{n\to\infty} \frac{H(s_\alpha^n|I)}{H(s_\alpha^n|G)},$$

is finite. □

PROOF Suppose $\overline{H} = \infty$, and take any subsequence along which the ratio of hazard rates diverges to infinity. Without loss of generality, suppose this is true of $\{s_\alpha^n\}$ itself. Note that the likelihood ratio along this sequence also diverges to infinity. We first claim that $s_\alpha^n \not\to \underline{S}$. If the sequence does converge to $\underline{S}$, note that $J_\alpha^n(s_\alpha^n, s_\alpha^n) = 0$ for high enough n, since the likelihood ratio has at most a finite number of discontinuity points by (A1). Now take any $0 < b < 1$, arbitrarily large c, and d satisfying $d > \frac{\alpha + c}{1 - \alpha}$. Then there exists m such that, for all $n > m$,

$$\frac{1 - F(s_\alpha^n|I)}{1 - F(s_\alpha^n|G)} \geq b \quad \text{and} \quad \frac{F(s_\alpha^n|I)}{F(s_\alpha^n|G)} \geq \left(\frac{1}{b}\right)^d,$$

where the inequalities follow from

$$\lim_{n\to\infty} \frac{1 - F(s_\alpha^n|I)}{1 - F(s_\alpha^n|G)} = 1 \text{ and } \lim_{n\to\infty} \frac{F(s_\alpha^n|I)}{F(s_\alpha^n|G)} = \lim_{n\to\infty} \frac{f(s_\alpha^n|I)}{f(s_\alpha^n|G)} = \infty,$$

respectively. Then, for all $n > m$,

$$\begin{aligned} \left(\frac{1-F(s_\alpha^n|I)}{1-F(s_\alpha^n|G)}\right)^{\alpha n-1}\left(\frac{F(s_\alpha^n|I)}{F(s_\alpha^n|G)}\right)^{n-\alpha n} &\geq b^{\alpha n-1}\left(\frac{1}{b}\right)^{d(n-\alpha n)} \\ &= \left(\frac{1}{b}\right)^{(d-d\alpha-\alpha)n+1} \\ &> \left(\frac{1}{b}\right)^{c}, \end{aligned}$$

where the last inequality follows from $d > \frac{\alpha+c}{1-\alpha}$. Since $b < 1$ and c is arbitrarily large, $J_\alpha^n(s_\alpha^n, s_\alpha^n) > 0$ for high enough n, a contradiction.

The remaining possibility is $s_\alpha^n \not\to \underline{S}$. Then there exists a subsequence lying above some $s' > \underline{S}$. For all $s'' \in (s', \overline{S})$, (A1) and (A3) imply

$$\sup_{s\in[s',s'']} \frac{H(s|I)}{H(s|G)} < \infty.$$

Since the ratio of hazard rates goes to infinity along the subsequence, the subsequence must converge to $\overline{S}$. But, applying L'Hôpital's rule, the ratio of hazard rates then converges to one along the subsequence, a contradiction. Therefore, $\alpha < 1$ implies $\overline{H}$ is finite. ■

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